On the Illumination of Three Dimensional Convex Bodies with Affine Plane Symmetry

Victoria Labute

September 11, 2015

Outline

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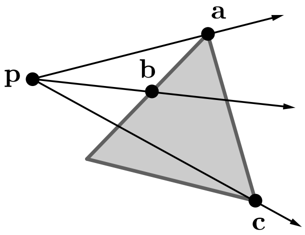

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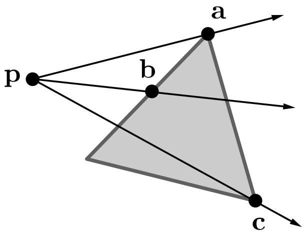

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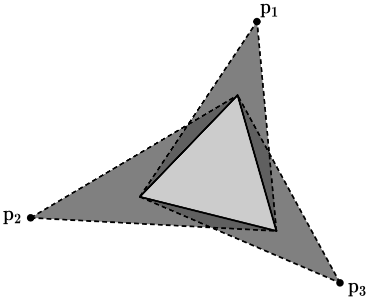

The minimum number of light sources needed to illuminate the triangle is three.

Surprisingly, the previous question is equivalent to the following question: for a convex body, what is the minimum number of smaller copies needed to cover it?

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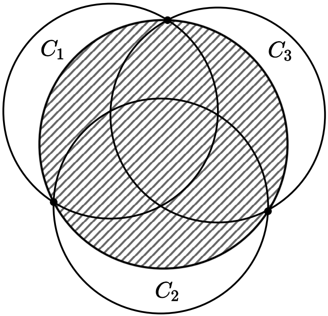
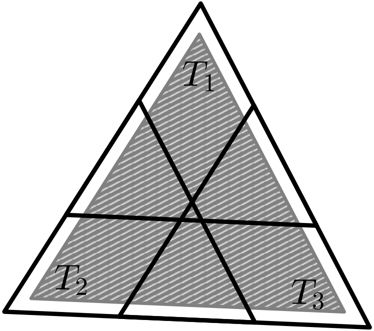


The minimum number of smaller discs required to cover the larger disc is 3.



The larger triangle can be covered by three smaller copies.

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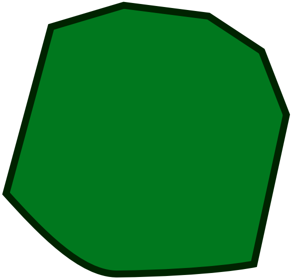
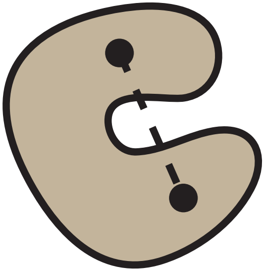
Examples:



Convex set



Non-convex set

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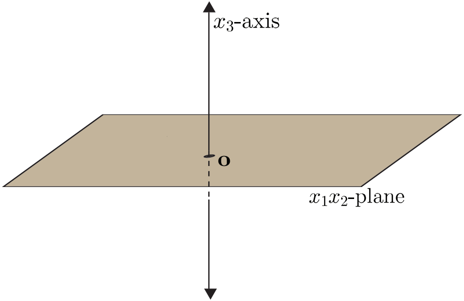

Unbounded sets, like planes, are not convex bodies

Outline

Let K be some convex body. A direction $\mathbf{d}$ is said to *illuminate* a point $\mathbf{x}$ on the boundary of K if the ray emanating from the point $\mathbf{x}$ with direction $\mathbf{d}$ intersects the interior of K.

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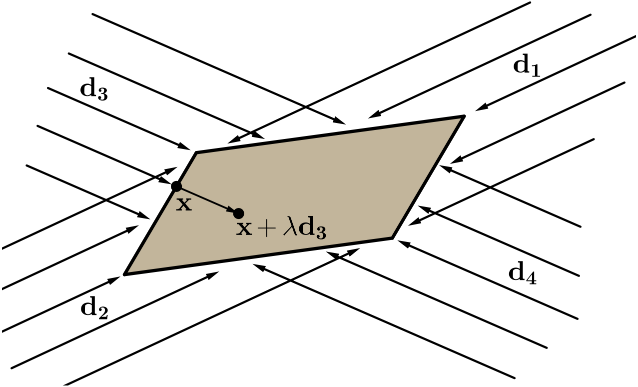

The point $\mathbf{x}$ on the boundary of the parallelogram is illuminated by the direction $\mathbf{d_3}$. The entire boundary of the parallelogram is illuminated by 4 directions.

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Boltyanski-Hadwiger Illumination Conjecture.

Every convex body K in $\mathbb{E}^n$ can be illuminated by 2^n external light sources or directions. If K is an affine n-cube, then exactly 2^n directions are required to illuminate K.

V. Boltyanski

H. Hadwiger

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  - Convex polyhedra in $\mathbb{E}^3$ with affine symmetry (for example, rotational symmetry and plane symmetry) can be illuminated by 2^3 directions (Bezdek, 1991).

A convex body K in $\mathbb{E}^3$ is said to be *affine plane symmetric* if there exists a line L and a plane H such that:

1. L meets H at exactly one point; and
2. for any $\mathbf{k} \in K$, there exists a vector $\mathbf{t} \in \mathbb{E}^3$ and a point $\mathbf{k}' \in K$ such that $[\mathbf{k}, \mathbf{k}'] \subseteq L + \mathbf{t}$ and $\frac{1}{2}(\mathbf{k} + \mathbf{k}') \in H \cap K$.

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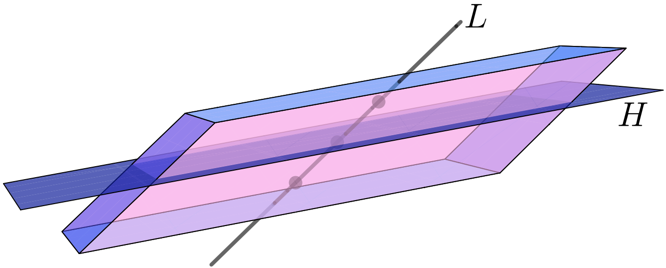

The parallelepiped is affine plane symmetric about the plane H with respect to the line L but H and L are not orthogonal.

Another partial result

Theorem (Dekster, 2000)

If $K \subset \mathbb{E}^3$ is an affine plane symmetric convex body, then K can be illuminated by eight directions.

B.V. Dekster

Outline

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- Denote the orthogonal projection of a convex body K onto the x_1x_2-plane by $\mathrm{Pr}(K) = B$; B will be referred to as the *base set* of K.

For example:

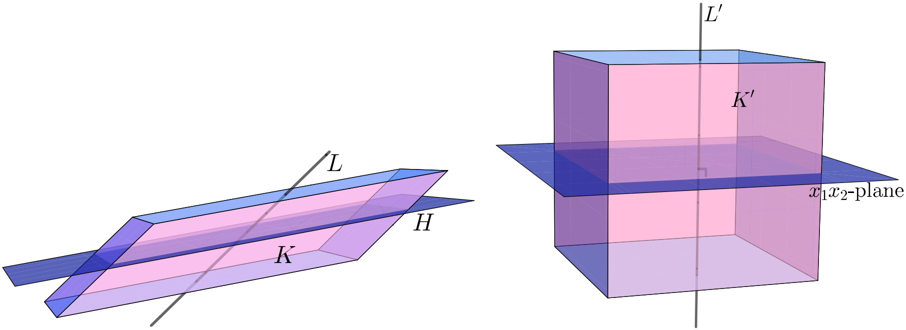

Figure : The parallelepiped, K, symmetric about H with respect to the line L is mapped by an affine transformation to the cube, K', symmetric about the x_1x_2-plane with respect to the line L', which is orthogonal to the x_1x_2-plane.

A point on the relative boundary of B is a *ground point* if $\mathrm{Pr}^{-1}(\mathbf{x}) \cap \mathrm{bd}(K) = \{\mathbf{x}\}$.

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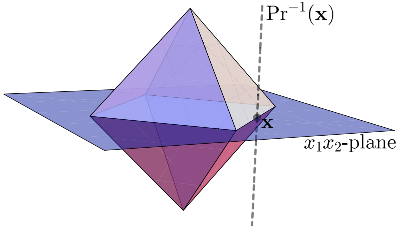
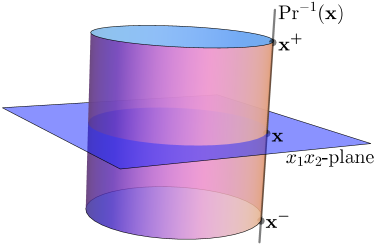


The point $\mathbf{x} \in \mathrm{relbd}(B)$ is a ground point. Each side of B is degenerate.



The point $\mathbf{x} \in \mathrm{relbd}(B)$ is a cliff point. All points on the relative boundary of B are cliff points

A hyperplane H *supports* a set C if

1. $H \cap C \neq \varnothing$; and
2. either $C \subseteq \overline{H}^{+}$ or $C \subseteq \overline{H}^{-}$.

Lemma.

Through each boundary point, $\mathbf{x}$*, of a closed, convex set* C *in* $\mathbb{E}^n$ *there passes at least one hyperplane supporting* C *at* $\mathbf{x}$*.*

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A boundary point of a closed convex set C is called *smooth* if there exists exactly one supporting hyperplane of C at that point.

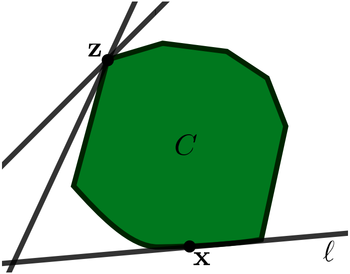


Figure : The point $\mathbf{x}$ is smooth. The point $\mathbf{z}$ is called a *singular point*; supporting lines through $\mathbf{z}$ are not unique.

Let **x** be some point on the relative boundary of B and let a ℓ be the supporting line of B at that point. Denote the supporting line of B parallel to ℓ by ℓ' . Then, each element of $\ell' \cap \text{relbd}(B)$ is called an *antipode* of **x**. The *complete antipode* of **x** is the set of its all antipodes; it will be denoted by $A(\mathbf{x})$.

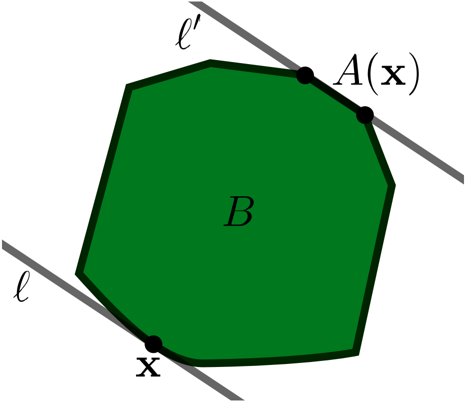


Figure : The complete antipode of the smooth point **x** is a line segment.

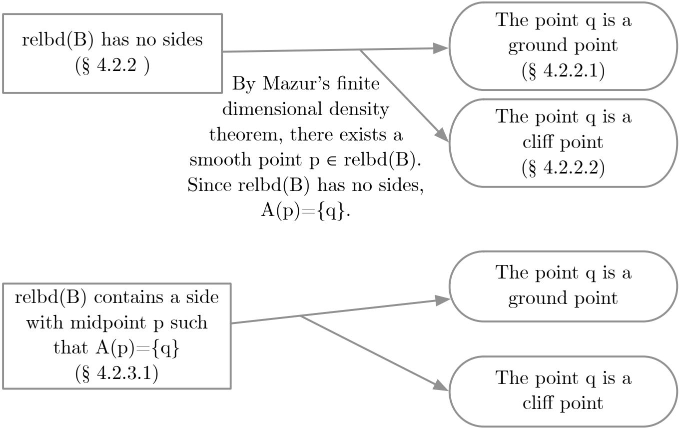
Two Important Theorems

Mazur's Finite Dimensional Density Theorem.
Smooth points are dense in the boundary of a convex body $K \subseteq \mathbb{E}^n$.

John-Löwner Theorem.
There exists a unique ellipsoid $\mathcal{E}$ of maximal volume contained in some convex body $K \subseteq \mathbb{E}^n$. Furthermore,

$$\mathcal{E} \subseteq K \subseteq n\,\mathcal{E}.$$

Cases of Dekster's proof



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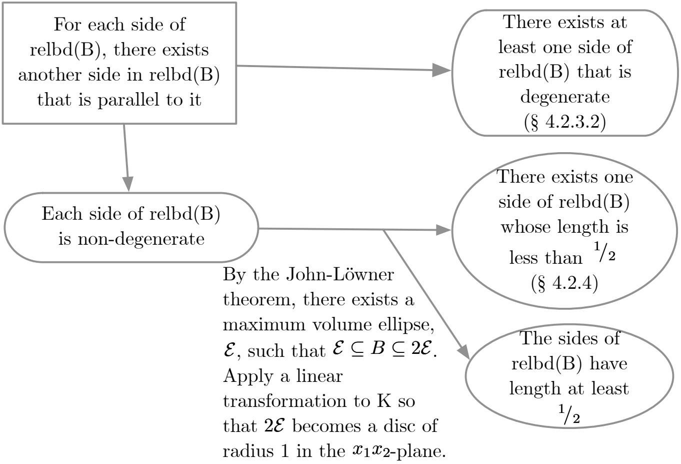

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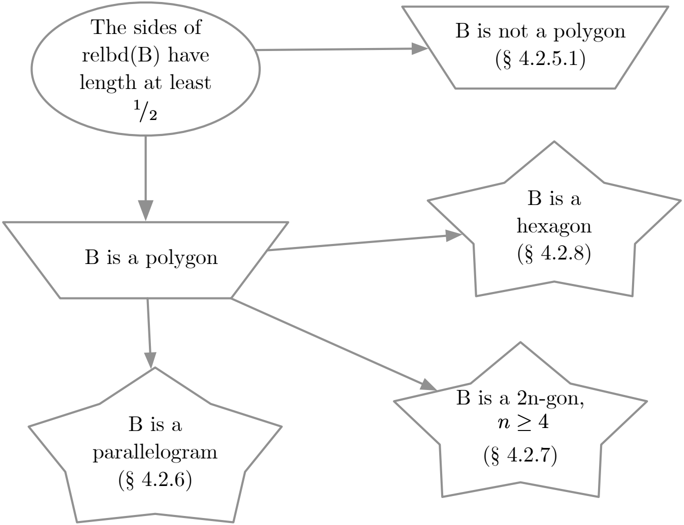

Outline

List of assumptions for this featured case

Suppose that

1. relbd(B) contains at least one side and for each side of relbd(B), there exists another side parallel to it;
2. each side of relbd(B) is non-degenerate;
3. each side of relbd(B) has length at least $\frac{1}{2}$; and
4. B is a hexagon.

This featured case breaks down into three further cases:

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Case 1: Suppose two consecutive vertices of B are cliff points.

Case 2: Suppose there exists a pair of parallel sides $[\mathbf{u}, \mathbf{v}]$ and $[\mathbf{w}, \mathbf{z}]$ of $\mathrm{relbd}(B)$ such that either $\mathbf{u}$ and $\mathbf{w}$ are ground points or $\mathbf{v}$ and $\mathbf{z}$ are ground points.

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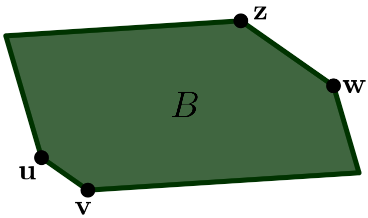

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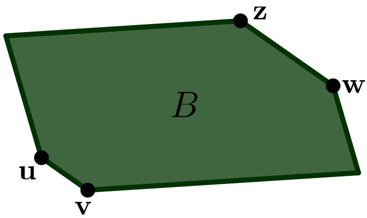

Case 3: Suppose the vertices of relbd(B) alternate between cliff and ground points.

Classifying the form of B

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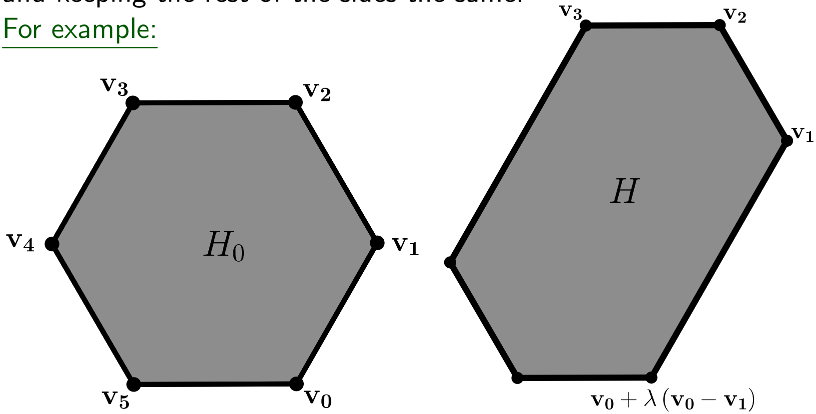

Figure : H is obtained from stretching the parallel sides $[\mathbf{v}_0, \mathbf{v}_1]$ and $[\mathbf{v}_3, \mathbf{v}_4]$ of H_0 by $\lambda \geqslant 1$.

Classifying the form of B continued

Proposition.

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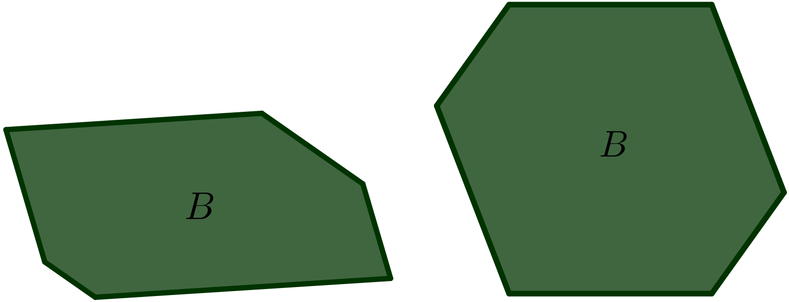
This means that

- Pairs of parallel sides for any affine image of H have the same length.
- Either B is an affine image of H or it is not.

Classifying the form of B continued



B is not affine image of H: at least one pair of parallel sides have different lengths

B is an affine image of H

Figure : Possible forms of the base set B in this case.

Third Case of the Featured Case

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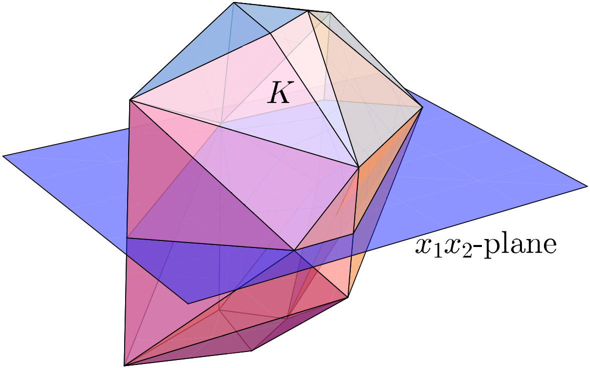

Figure : The base set of K is a hexagon that satisfies the following four conditions: it is not an affine image of H; its vertices alternate between ground and cliff points; each of its sides are non-degenerate; and for each side, there exists another parallel to it.

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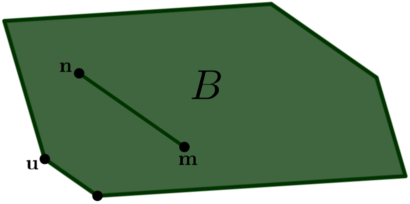
Lemma (4.2.8.3.2)

Let B be a hexagon such that

1. for any side of B, there exists another side of B parallel to it;
2. B is not the affine image of a hexagon obtained by scaling the lengths of exactly one pair of parallel sides from a regular hexagon by a scalar $\lambda \geqslant 0$ while preserving the other edge lengths.

Then, $\mathrm{relint}(B)$ contains a line segment $[\mathbf{n}, \mathbf{m}]$ such that $\mathbf{m} - \mathbf{n} = 2\,(\mathbf{v} - \mathbf{u})$, for some side $[\mathbf{u}, \mathbf{v}] \subseteq \mathrm{relbd}(B)$.



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Suppose B is not an affine image of H. Then, by Lemma 4.2.8.3.2, there exists a side $[\mathbf{u}, \mathbf{v}] \subseteq \text{relbd}(B)$ such that $[\mathbf{m}, \mathbf{n}] \subseteq \text{relint}(B)$.

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$$\left\|\mathbf{k}^+ - \mathbf{k}^-\right\| = \max\left\{\left\|\mathbf{f}^+ - \mathbf{f}^-\right\| \mid \text{for all cliff points } \mathbf{f} \in [\mathbf{u}, \mathbf{v}]\right\}.$$

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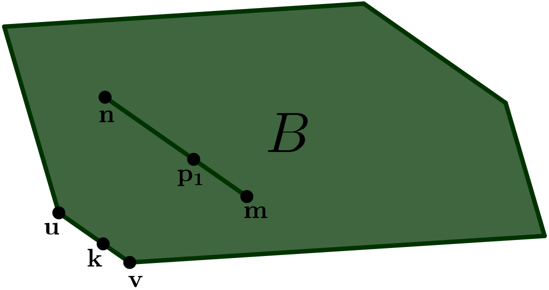
Let $\mathbf{p_1} \in [\mathbf{m}, \mathbf{n}]$ be chosen so that $\mathbf{p_1} - \mathbf{n} = 2(\mathbf{k} - \mathbf{u})$ and $\mathbf{m} - \mathbf{p_1} = 2(\mathbf{v} - \mathbf{k})$.



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Suppose B is not an affine image of H. Then, by Lemma 4.2.1.2, the directions $(\mathbf{p_1} - \mathbf{k}) - (\mathbf{k}^+ - \mathbf{k})$ and $(\mathbf{p_1} - \mathbf{k}) - (\mathbf{k}^- - \mathbf{k})$ illuminate $W_{[\mathbf{u},\mathbf{v}]}$.

Third Case of the Featured Case

Suppose B is not an affine image of H. Then, by Lemma 4.2.1.2, the directions $(\mathbf{p_1} - \mathbf{k}) - (\mathbf{k}^{+} - \mathbf{k})$ and $(\mathbf{p_1} - \mathbf{k}) - (\mathbf{k}^{-} - \mathbf{k})$ illuminate $W_{[\mathbf{u},\mathbf{v}]}$.
By Proposition 4.2.2.2.2, there exists a real number $\chi' > 0$ such that the directions $(\mathbf{p_1} - \mathbf{k}) - (\mathbf{k}^{+} - \mathbf{k})$ and $(\mathbf{p_1} - \mathbf{k}) - (\mathbf{k}^{-} - \mathbf{k})$ illuminate an open neighbourhood of $W_{[\mathbf{u},\mathbf{v}]}$ on the boundary of K, $W_{[\mathbf{u},\mathbf{v}]} + \chi' \mathrm{B}(\mathbf{o}, 1)$.

Third Case of the Featured Case

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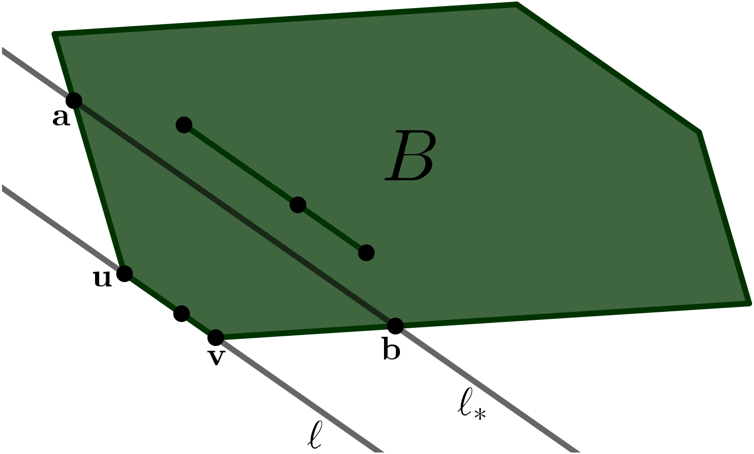
By Proposition 4.2.2.2.2, there exists a real number $\chi' > 0$ such that the directions $(\mathbf{p_1} - \mathbf{k}) - (\mathbf{k}^+ - \mathbf{k})$ and $(\mathbf{p_1} - \mathbf{k}) - (\mathbf{k}^- - \mathbf{k})$ illuminate an open neighbourhood of $W_{[\mathbf{u},\mathbf{v}]}$ on the boundary of K, $W_{[\mathbf{u},\mathbf{v}]} + \chi' \mathrm{B}(\mathbf{o}, 1)$.

It follows from Lemma 4.2.2.1.3, Proposition 4.2.2.1.8 and Proposition 4.2.2.1.9 that there exists points **a** and **b** in this open neighbourhood such that the line between them is parallel to the supporting line at the side $[\mathbf{u}, \mathbf{v}]$.


a
B
u
v
b
ℓ_*
ℓ

Third Case of the Featured Case

Suppose B is not an affine image of H. Let $\mathbf{q} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$.

Third Case of the Featured Case

Suppose B is not an affine image of H. Let $\mathbf{q} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$. Then, by Lemma 4.2.2.1.12, the six directions

- $\frac{1}{2}(\mathbf{q} + \mathbf{a}) - \mathbf{b}$,
- $\frac{1}{2}(\mathbf{q} + \mathbf{b}) - \mathbf{a}$,
- $\left(1 - \frac{2\xi}{1+\xi}\right)\left(\frac{1}{2}(\mathbf{q} + \mathbf{a}) - \mathbf{b}\right) \pm \mathcal{T}\mathbf{e}_3$, and
- $\left(1 - \frac{2(1-\xi)}{2-\xi}\right)\left(\frac{1}{2}(\mathbf{q} + \mathbf{b}) - \mathbf{a}\right) \pm \mathcal{T}\mathbf{e}_3$

will illuminate $\mathrm{bd}(K) \backslash \left(W_{[\mathbf{u},\mathbf{v}]} + \chi' \mathrm{B}(\mathbf{o}, 1)\right)$.